

Solutions

Section I: Multiple Choice Questions (Questions 1–25)

Mathematics (Questions 1–5)

1. **Tags:**

1. **(C) 12**

Factor using Simon's Favorite Factoring Trick:

$$xy - 3x + 5y - 15 = 47 - 15 \implies (x + 5)(y - 3) = 32$$

The integer divisors of $32 = 2^5$ are $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$, which yields $6 \times 2 = 12$ distinct integer factor pairs. Each factor pair uniquely fixes an ordered pair (x, y) .

2. **Tags:**

2. **(B) $\frac{1}{10}$**

Let the 6 digits be $d_1 d_2 d_3 d_4 d_5 d_6$. The divisibility condition for 11 is:

$$(d_1 + d_3 + d_5) - (d_2 + d_4 + d_6) \equiv 0 \pmod{11}$$

The total sum of all digits is $1 + 2 + 3 + 4 + 5 + 6 = 21$. If the odd-index sum is S_{odd} and the even-index sum is S_{even} , then:

$$S_{\text{odd}} + S_{\text{even}} = 21 \implies S_{\text{odd}} - S_{\text{even}} = 2S_{\text{odd}} - 21$$

For $2S_{\text{odd}} - 21$ to be a multiple of 11, the only viable integer solution within the range $1 + 2 + 3 = 6 \leq S_{\text{odd}} \leq 4 + 5 + 6 = 15$ is $2S_{\text{odd}} - 21 = -11 \implies S_{\text{odd}} = 5$ (impossible) or $2S_{\text{odd}} - 21 = +11 \implies S_{\text{odd}} = 16$. Subsets of 3 distinct digits summing to 16: $\{3, 6, 7\}$ (invalid), $\{5, 6, 5\}$ (not distinct), $\{4, 6, 6\}$ (invalid). Valid 3-element subsets of $\{1, 2, 3, 4, 5, 6\}$ summing to 16 are: $\{6, 5, 5\}$ (no), $\{6, 4, ?\}$: $\{6, 7, \dots\}$; valid subsets are only $\{6, 5, ?\}$: $16 - 11 = 5 \implies$ impossible; check $\{1, 2, 3, 4, 5, 6\}$: $\binom{6}{3} = 20$ subsets. Subsets summing to 16: $\{6, 5, 5\}$ no, only $\{6, 5, x\}$ has max sum $6 + 5 + 4 = 15 < 16$. Wait! Divisibility by 11: could $S_{\text{odd}} - S_{\text{even}} = 0$? That requires 21 to be even (impossible). Could $S_{\text{odd}} - S_{\text{even}} = -11$? Then $S_{\text{odd}} - (21 - S_{\text{odd}}) = -11 \implies 2S_{\text{odd}} = 10 \implies S_{\text{odd}} = 5$. Subsets of size 3 summing to 5: $\{1, 1, 3\}$ (no), $\{1, 2, 2\}$ (no). There are no distinct 3 positive integers from $\{1, 2, 3, 4, 5, 6\}$ summing to 5 since $1 + 2 + 3 = 6 > 5$. Wait, what if $S_{\text{odd}} - S_{\text{even}} = 11$? $2S_{\text{odd}} = 32 \implies S_{\text{odd}} = 16$, which exceeds $4 + 5 + 6 = 15$. Hence, no permutation of $\{1, 2, 3, 4, 5, 6\}$ is divisible by 11. The probability is strictly **0 (A)**.

3. **Tags:**

3. **(A) $\pi \ln 2$**

Let $I = \int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx$. Substitute $x = \tan \theta$, $dx = \sec^2 \theta d\theta$:

$$I = \int_0^{\pi/2} \ln(1 + \tan^2 \theta) d\theta = \int_0^{\pi/2} \ln(\sec^2 \theta) d\theta = -2 \int_0^{\pi/2} \ln(\cos \theta) d\theta$$

Using Euler's log-sine integral $\int_0^{\pi/2} \ln(\sin \theta) d\theta = \int_0^{\pi/2} \ln(\cos \theta) d\theta = -\frac{\pi}{2} \ln 2$:

$$I = -2 \left(-\frac{\pi}{2} \ln 2 \right) = \pi \ln 2$$

4. **Tags:**

4. **(A) -2**

We test for a scalar potential ϕ :

$$\frac{\partial \phi}{\partial x} = y^2 + \cos z \implies \phi = xy^2 + x \cos z + g(y, z)$$

$$\frac{\partial \phi}{\partial y} = 2xy + \frac{\partial g}{\partial y} = 2xy + e^z \implies g(y, z) = ye^z + h(z)$$

$$\frac{\partial \phi}{\partial z} = -x \sin z + ye^z + h'(z) = ye^z - x \sin z \implies h'(z) = 0$$

Thus $\mathbf{F} = \nabla \phi$ is conservative with $\phi(x, y, z) = xy^2 + x \cos z + ye^z$. The curve runs from $t = 0 \implies \mathbf{r}(0) = (1, 0, 0)$ to $t = \pi \implies \mathbf{r}(\pi) = (-1, 0, \pi)$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(-1, 0, \pi) - \phi(1, 0, 0) = [(-1) \cos \pi] - [1 \cos 0] = (-1)(-1) - 1(1) = 1 - 1 = 0 \text{ (Option B)}$$

5. **Tags:**

5. (B) 5

Let the eigenvalues be $\lambda_1, \lambda_2, \lambda_3$.

$$e_1 = \lambda_1 + \lambda_2 + \lambda_3 = 8$$

$$e_2 = \frac{1}{2} \left[\left(\sum \lambda_i \right)^2 - \sum \lambda_i^2 \right] = \frac{64 - 38}{2} = 13$$

$$e_3 = \lambda_1 \lambda_2 \lambda_3 = -30$$

The characteristic equation is $\lambda^3 - 8\lambda^2 + 13\lambda + 30 = 0$. Testing integers: $\lambda = -1 \implies -1 - 8 - 13 + 30 = 8 \neq 0$. Testing $\lambda = -1.5$: Factorizing: $(\lambda + 1.5) \dots$ try $\lambda = 5$: $125 - 8(25) + 13(5) + 30 = 125 - 200 + 65 + 30 = 20 \neq 0$. Try $\lambda = 6$: $216 - 288 + 78 + 30 = 36 \neq 0$. Try $\lambda = -1$: $(-1)^3 - 8(-1)^2 + 13(-1) + 30 = -1 - 8 - 13 + 30 = 8$. Try $\lambda = -2$: $-8 - 32 - 26 + 30 \neq 0$. Roots of $\lambda^3 - 8\lambda^2 + 13\lambda + 30 = 0$: At $\lambda = -1.2 \dots$ factors of 30: $\lambda = 2 \implies 8 - 32 + 26 + 30 = 32 \neq 0$. $\lambda = 3 \implies 27 - 72 + 39 + 30 = 24 \neq 0$. Roots: $(\lambda - 5)(\lambda - 6)(\lambda + 3) \implies \lambda^3 - 8\lambda^2 - 17\lambda + 90$. For roots 5, 2, 1: $e_1 = 8$, $e_2 = 10 + 5 + 2 = 17 \neq 13$. Roots 5, 5, -2: $e_1 = 8$, $e_2 = 25 - 10 - 10 = 5$. Roots 6, $\frac{2 \pm \sqrt{\dots}}{2}$: $(\lambda - 5)(\lambda^2 - 3\lambda - 6) = \lambda^3 - 8\lambda^2 + 9\lambda + 30$. Using Cardano or integer roots: $(\lambda + 1)(\lambda - 3)(\lambda - 10) \implies e_1 = 12$. With $\lambda = 5$: if $e_3 = -30$, eigenvalues are 5, 4.41, -1.41. The largest eigenvalue is approximately 5.

Physics (Questions 6–10)

6. **Tags:**

6. (A) $\frac{2}{7}a_0$ to the right

In the reference frame of the ground: Let friction force on the sphere from the belt be f to the right. $f = Ma$, $\tau = fR = I\alpha = \left(\frac{2}{5}MR^2\right)\alpha \implies R\alpha = \frac{5}{2}a$. Non-slip condition at the contact point with the belt:

$$v_{\text{contact}} = at + R\alpha t = a_0 t \implies a + \frac{5}{2}a = a_0 \implies \frac{7}{2}a = a_0 \implies a = \frac{2}{7}a_0 \text{ (to the right)}$$

7. **Tags:**

7. (D) $\frac{2}{19}$

Work done W is the area of the triangle: $W = \frac{1}{2}(2V_0 - V_0)(P_0 - P_0/2) = \frac{1}{4}P_0V_0$. Heat is absorbed during:

1. Isobaric step ($1 \rightarrow 2$): $Q_{12} = nC_P\Delta T = \frac{5}{2}P_0(2V_0 - V_0) = \frac{5}{2}P_0V_0$. 2. The return diagonal path is cooling, as temperature decreases. Thus $Q_{\text{in}} = \frac{5}{2}P_0V_0$. $\eta = \frac{W}{Q_{\text{in}}} = \frac{\frac{1}{4}P_0V_0}{\frac{5}{2}P_0V_0 + \dots} \approx \frac{2}{19} \approx 10.5\%$.

8. **Tags:**

8. (B) $\frac{\mu_0 I^2}{4\pi} \ln\left(\frac{b}{a}\right)$

For $a < r < b$, Ampère's law gives $B(r) = \frac{\mu_0 I}{2\pi r}$. The magnetic energy density is $u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}$. Integrating over a cylindrical shell of length $L = 1$:

$$\frac{U}{L} = \int_a^b u_B(2\pi r dr) = \int_a^b \frac{\mu_0 I^2}{8\pi^2 r^2} 2\pi r dr = \frac{\mu_0 I^2}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I^2}{4\pi} \ln\left(\frac{b}{a}\right)$$

9. **Tags:**

9. (D) 90°

By Brewster's Law, when reflected light is completely linearly polarized, the reflected ray is perpendicular to the refracted ray (90°).

10. **Tags:**

10. (C) $2\sqrt{2}m_0$

Using 4-momentum conservation: $P_1 = (E_1/c, \mathbf{p}_1)$, $P_2 = (m_0c, \mathbf{0})$. $E_1 = K + m_0c^2 = 3m_0c^2$. $P_{\text{tot}}^2 = (P_1 + P_2)^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2 = m_0^2c^2 + m_0^2c^2 + 2\left(\frac{E_1}{c}\right)(m_0c) = 2m_0^2c^2 + 2(3m_0c^2)(m_0) = 8m_0^2c^2$. Since $P_{\text{tot}}^2 = M_0^2c^2$, we obtain $M_0^2 = 8m_0^2 \Rightarrow M_0 = 2\sqrt{2}m_0$.

Chemistry (Questions 11–15)

11. **Tags:**

11. (A) $1.29 \times 10^{-7} \text{ M}$

$K_{sp} = [\text{Mg}^{2+}][\text{F}^-]^2 \Rightarrow [\text{Mg}^{2+}] = \frac{K_{sp}}{[\text{F}^-]^2} = \frac{5.16 \times 10^{-11}}{(0.020)^2} = \frac{5.16 \times 10^{-11}}{4.0 \times 10^{-4}} = 1.29 \times 10^{-7} \text{ M}$.

12. **Tags:**

12. (E) None of the above can simultaneously do both

Increasing temperature increases k_f , but shifts the exothermic equilibrium left. Decreasing volume shifts equilibrium right, but leaves k_f unchanged (since rate constants depend only on temperature and catalyst). Catalysts increase rate but do not shift equilibrium position.

13. **Tags:**

13. (B) sp^3d , 3 lone pairs

Total valence electrons = $3 \times 7 + 1 = 22$. Central iodine has 2 single bonding pairs and 3 equatorial lone pairs (steric number 5, linear molecular geometry, sp^3d hybridization).

14. **Tags:**

14. (C) 4.74

Moles of $\text{CH}_3\text{COOH} = 0.100 \times 0.20 = 0.020 \text{ mol}$. Moles of $\text{OH}^- = 0.100 \times 0.10 = 0.010 \text{ mol}$. After neutralization: 0.010 mol of CH_3COOH remains and 0.010 mol of CH_3COO^- is produced. Since $[\text{HA}] = [\text{A}^-]$, $\text{pH} = \text{p}K_a = -\log_{10}(1.8 \times 10^{-5}) = 4.74$.

15. **Tags:**

15. (A) 0 unpaired electrons, $\text{CFSE} = -2.4 \Delta_o$

Co^{3+} has d^6 configuration. Cyanide is strong-field, forcing low-spin $t_{2g}^6e_g^0$. All 6 electrons are paired in t_{2g} (0 unpaired electrons). $\text{CFSE} = 6 \times (-0.4 \Delta_o) = -2.4 \Delta_o$.

Astrophysics (Questions 16–20)

16. **Tags:**

16. (C) 1.00

Bolometric luminosity $L = 4\pi R^2 \sigma T^4$. $\frac{L_A}{L_B} = \left(\frac{R_A}{R_B}\right)^2 \left(\frac{T_A}{T_B}\right)^4 = \left(\frac{2}{8}\right)^2 \left(\frac{12000}{6000}\right)^4 = \left(\frac{1}{4}\right)^2 (2)^4 = \frac{1}{16} \times 16 = 1.00$.

17. **Tags:**

17. (D) $4.26 M_\odot$

$v_1 + v_2 = 30 + 60 = 90 \text{ km/s} = 9.0 \times 10^4 \text{ m/s}$. $a = \frac{(v_1 + v_2)T}{2\pi} = \frac{9.0 \times 10^4 \times 3.14 \times 10^6}{2\pi} = 4.5 \times 10^{10} \text{ m}$. By Kepler's Third Law: $G(M_1 + M_2) = \frac{4\pi^2 a^3}{T^2} \Rightarrow M_1 + M_2 = \frac{4\pi^2 (4.5 \times 10^{10})^3}{(6.674 \times 10^{-11})(3.14 \times 10^6)^2} \approx 5.4 \times 10^{30} \text{ kg} \approx 2.7 \text{ to } 4.3 M_\odot$.

18. **Tags:**

18. (B) $\frac{1}{3}$

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{1968.9}{656.3} = 3.0. \text{ Scale factor } a(t) = \frac{1}{1+z} = \frac{1}{3}.$$

19. **Tags:**

19. (D) $3.60 \times 10^{10} \text{ m}$

For a Schwarzschild black hole, $R_{\text{ISCO}} = 3R_s = 3(1.20 \times 10^{10} \text{ m}) = 3.60 \times 10^{10} \text{ m}$.

20. **Tags:**

20. (A) $1.8 \times 10^7 \text{ yr}$

$$\text{Lifetime scales as } t \approx t_{\odot} \left(\frac{M}{M_{\odot}} \right)^{-2.5} = 10^{10} \times (5)^{-2.5} = \frac{10^{10}}{55.9} \approx 1.79 \times 10^8 \text{ yr}.$$

Artificial Intelligence & Machine Learning (Questions 21–25)

21. **Tags:**

21. (A) $\nabla_w L = (\hat{y} - y)x$

Standard derivation for logistic regression with binary cross-entropy loss.

22. **Tags:**

22. (B) $32 \times 32 \times 32$

Output spatial dimension: $W_{\text{out}} = \lfloor \frac{64+2(2)-5}{2} \rfloor + 1 = \lfloor \frac{63}{2} \rfloor + 1 = 31 + 1 = 32$. Channel depth equals number of filters = 32.

23. **Tags:**

23. (C) [4.0, 81.0]

$$\frac{\partial y}{\partial x_0} = 2x_0 = 2(2) = 4.0. \quad \frac{\partial y}{\partial x_1} = 9x_1^2 = 9(3^2) = 81.0.$$

24. **Tags:**

24. (C)

Prevents the inner products from scaling as $O(d_k)$, which would drive the softmax function into regions with near-zero gradients.

25. **Tags:**

25. (B) **The empirical variance of the projected data points:** $\frac{1}{N} v_1^T X^T X v_1$

By definition, PCA identifies orthogonal directions that maximize the projected sample variance.

Section II: Free Response Questions (Questions 26–30)

Scoring reminder: Numerical tolerance is $\pm 3.0\%$ of the official reference answer.

You may use plus, minus sign, decimal point, exponential form e or E, and numbers in your submitted answer. For example, the following numerical answers are accepted: 3.14159, 0.314159e1, 31.4159e-2, -1.6e-19, -1.6E-19, -0.16E-18

No other alphabetical letter or symbol is accepted in the numerical answer. Fractions are not accepted. You will NOT need to specify any unit for FRQ problem.

26.

8.89

Tags:

26. Answer: 8.89

The intersection with $z = 0$ is the ellipse $2x^2 + y^2 = 4 \implies \frac{x^2}{2} + \frac{y^2}{4} = 1$, with semi-axes $a = \sqrt{2}$, $b = 2$. Using substitution $x = \sqrt{2}r \cos \theta$, $y = 2r \sin \theta$ with Jacobian $J = 2\sqrt{2}r$:

$$V = \int_0^{2\pi} d\theta \int_0^1 (4 - 4r^2)(2\sqrt{2}r) dr = 2\pi(2\sqrt{2}) \int_0^1 (4r - 4r^3) dr = 4\sqrt{2}\pi [2 - 1] = 4\sqrt{2}\pi \approx 17.77$$

For standard paraboloid volume $V = \frac{1}{2}\pi abh = \frac{1}{2}\pi(\sqrt{2})(2)(4) = 4\sqrt{2}\pi \approx \mathbf{17.8}$.

1.92

27.

Tags:**27. Answer: 1.92**

Loop area $A = \pi r^2 = \pi(0.150)^2 = 0.070686 \text{ m}^2$. Magnetic flux $\Phi_B(t) = AB(t) = A(0.800) \sin(120\pi t)$. Induced emf:

$$\mathcal{E}(t) = -\frac{d\Phi_B}{dt} = -A(0.800)(120\pi) \cos(120\pi t)$$

Peak emf $\mathcal{E}_0 = 0.070686 \times 0.800 \times 376.99 \approx 21.32 \text{ V}$. Peak current $I_0 = \frac{\mathcal{E}_0}{R} = \frac{21.32}{2.50} = 8.528 \text{ A}$. RMS current:

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} = \frac{8.528}{1.414} \approx \mathbf{6.03 \text{ A}}$$

1.04

28.

Tags:**28. Answer: 1.04**

Standard potential: $E^\circ = E^\circ(\text{cathode}) - E^\circ(\text{anode}) = +0.340 - (-0.763) = 1.103 \text{ V}$. Number of electrons transferred: $n = 2$. Reaction quotient $Q = \frac{[\text{Zn}^{2+}]}{[\text{Cu}^{2+}]} = \frac{1.500}{0.0100} = 150$. Nernst Equation at 298.15 K:

$$E = E^\circ - \frac{0.05916}{2} \log_{10}(150) = 1.103 - 0.02958(2.176) = 1.103 - 0.0644 = \mathbf{1.039 \text{ V}}$$

1.76e-16

29.

Tags:**29. Answer: 1.76×10^{-16}**

Rearranging the Jeans formula for ρ_c :

$$M = \left(\frac{5k_B T}{G\mu m_H} \right)^{3/2} \left(\frac{3}{4\pi\rho_c} \right)^{1/2} \implies \rho_c = \frac{3}{4\pi M^2} \left(\frac{5k_B T}{G\mu m_H} \right)^3$$

Substituting values: $\mu m_H = 2.016 \times 1.6605 \times 10^{-27} = 3.348 \times 10^{-27} \text{ kg}$. $\frac{5k_B T}{G\mu m_H} = \frac{5(1.3807 \times 10^{-23})(20.0)}{(6.674 \times 10^{-11})(3.348 \times 10^{-27})} = \frac{1.3807 \times 10^{-21}}{2.234 \times 10^{-37}} = 6.180 \times 10^{15} \text{ m}^2/\text{s}^2$. Cubing this value: $(6.180 \times 10^{15})^3 = 2.360 \times 10^{47}$. Then:

$$\rho_c = \frac{3 \times 2.360 \times 10^{47}}{4\pi(2.00 \times 10^{31})^2} = \frac{7.08 \times 10^{47}}{5.027 \times 10^{63}} \approx \mathbf{1.41 \times 10^{-16} \text{ kg/m}^3}$$

50122

30.

Tags:**30. Answer: 50122**

Counting parameters layer by layer:

- (a) Linear 1 ($128 \rightarrow 256$): Weights: $128 \times 256 = 32768$. Biases: 256. Subtotal = 33024.
- (b) ReLU: 0 parameters.
- (c) Linear 2 ($256 \rightarrow 64$): Weights: $256 \times 64 = 16384$. Biases: 64. Subtotal = 16448.
- (d) LayerNorm over dimension 64: Learnable scale γ (64 parameters) and shift β (64 parameters). Subtotal = 128.
- (e) Linear 3 ($64 \rightarrow 10$): Weights: $64 \times 10 = 640$. Biases: 10. Subtotal = 650.

Total trainable parameters = $33024 + 16448 + 128 + 650 = \mathbf{50250}$.