

Section I: Multiple Choice Questions (Questions 1–25)**Mathematics (Questions 1–5)**

1. Find the number of ordered pairs of integers (x, y) satisfying the Diophantine equation:

$$xy - 3x + 5y = 47$$

- (A) 4
(B) 8
(C) 12
(D) 16
(E) 24
2. Let S denote the set of all 6-digit integers formed by permuting the digits $\{1, 2, 3, 4, 5, 6\}$ without repetition. If one integer N is selected uniformly at random from S , what is the probability that N is divisible by 11?

- (A) 0
(B) $\frac{1}{10}$
(C) $\frac{1}{11}$
(D) $\frac{3}{20}$
(E) $\frac{1}{6}$
3. Evaluate the improper definite integral:

$$I = \int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2} dx$$

- (A) $\pi \ln 2$
(B) $\frac{\pi}{2} \ln 2$
(C) $\pi \ln 4$
(D) $\frac{\pi^2}{8}$
(E) $\frac{\pi}{4} \ln 2$
4. Consider the vector field $\mathbf{F}(x, y, z) = (y^2 + \cos z) \mathbf{i} + (2xy + e^z) \mathbf{j} + (ye^z - x \sin z) \mathbf{k}$. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the smooth space curve C parameterized by $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $t \in [0, \pi]$.
- (A) -2
(B) 0

- (C) 1
- (D) 2
- (E) π

5. Let M be a real symmetric 3×3 matrix with $\text{Tr}(M) = 8$, $\text{Tr}(M^2) = 38$, and $\det(M) = -30$. What is the largest eigenvalue of M ?

- (A) 3
- (B) 5
- (C) 6
- (D) 8
- (E) 10

Physics (Questions 6–10)

6. A uniform solid sphere of mass M and radius R is placed on a rough horizontal conveyor belt that accelerates horizontally to the right with constant acceleration a_0 . Assuming the sphere rolls without slipping on the belt surface, what is the magnitude and direction of the linear acceleration a of the sphere's center of mass relative to the stationary ground?

- (A) $\frac{2}{7}a_0$ to the right
- (B) $\frac{2}{7}a_0$ to the left
- (C) $\frac{5}{7}a_0$ to the right
- (D) $\frac{5}{7}a_0$ to the left
- (E) $\frac{3}{5}a_0$ to the right

7. One mole of a monatomic ideal gas ($C_V = \frac{3}{2}R$) undergoes a reversible thermodynamic cycle consisting of an isobaric expansion from (P_0, V_0) to $(P_0, 2V_0)$, an isochoric depressurization to $(P_0/2, 2V_0)$, and a linear path in the P – V plane returning back to (P_0, V_0) . What is the thermal efficiency η of this heat engine?

- (A) $\frac{1}{13}$
- (B) $\frac{2}{17}$
- (C) $\frac{1}{11}$
- (D) $\frac{2}{19}$
- (E) $\frac{3}{26}$

8. A long coaxial cable consists of an inner solid cylinder of radius a carrying uniform current I in the $+z$ direction, and a thin concentric outer cylindrical shell of radius $b > a$ carrying equal return current I distributed uniformly in the $-z$ direction. What is the total magnetic energy per unit length stored in the region $a < r < b$?
- (A) $\frac{\mu_0 I^2}{8\pi}$
(B) $\frac{\mu_0 I^2}{4\pi} \ln\left(\frac{b}{a}\right)$
(C) $\frac{\mu_0 I^2}{2\pi} \left(\frac{b-a}{a}\right)$
(D) $\frac{\mu_0 I^2}{8\pi} \ln\left(\frac{b}{a}\right)$
(E) $\frac{\mu_0 I^2}{4\pi} \left(\frac{b^2 - a^2}{a^2}\right)$
9. An unpolarized light beam in air is incident on the planar boundary of a dielectric slab of refractive index $n = \sqrt{3}$. The angle of incidence is adjusted such that the reflected beam is completely linearly polarized. What is the angle between the reflected ray and the refracted ray?
- (A) 30°
(B) 45°
(C) 60°
(D) 90°
(E) 120°
10. A particle of rest mass m_0 and kinetic energy $K = 2m_0c^2$ collides head-on with an identical particle of rest mass m_0 that is initially at rest in the laboratory frame. If the two particles fuse into a single composite particle during the inelastic collision, what is the rest mass M_0 of the resulting product?
- (A) $2m_0$
(B) $\sqrt{6}m_0$
(C) $2\sqrt{2}m_0$
(D) $3m_0$
(E) $\sqrt{10}m_0$

Chemistry (Questions 11–15)

11. An aqueous solution is saturated with magnesium fluoride (MgF_2) at 25°C where $K_{sp} = 5.16 \times 10^{-11}$. If solid sodium fluoride (NaF) is dissolved into this solution until $[\text{F}^-] = 0.020 \text{ M}$ (assume volume changes are negligible), what is the resulting equilibrium concentration of Mg^{2+} ions?
- (A) $1.29 \times 10^{-7} \text{ M}$

- (B) 2.58×10^{-7} M
- (C) 5.16×10^{-7} M
- (D) 1.29×10^{-9} M
- (E) 3.72×10^{-4} M

12.



Which of the following simultaneous actions will shift the equilibrium towards the right (increasing product yield) and increase the equilibrium rate constant k_f of the forward reaction?

- (A) Decreasing the reaction vessel volume at constant temperature
- (B) Increasing the temperature at constant volume
- (C) Adding an inert gas at constant volume
- (D) Adding an effective heterogeneous catalyst at constant temperature and pressure
- (E) None of the above can simultaneously do both

13. What is the hybridization of the central atom and the total number of lone electron pairs on the central atom in the triiodide anion (I_3^-)?

- (A) sp^3 , 2 lone pairs
- (B) sp^3d , 3 lone pairs
- (C) sp^3d^2 , 4 lone pairs
- (D) sp^2 , 1 lone pair
- (E) sp^3d , 2 lone pairs

14. A buffer solution is prepared by mixing 100 mL of 0.20 M acetic acid (CH_3COOH , $K_a = 1.8 \times 10^{-5}$) with 100 mL of 0.10 M sodium hydroxide (NaOH). What is the final pH of this buffer solution at 25°C ? ($\log_{10}(1.8) \approx 0.255$)

- (A) 3.75
- (B) 4.44
- (C) 4.74
- (D) 5.04
- (E) 5.25

15. In transition metal coordination chemistry, the complex ion $[\text{Co}(\text{CN})_6]^{3-}$ has 6 electrons in the cobalt $3d$ subshell. Given that CN^- is a strong-field ligand causing low-spin octahedral splitting, how many unpaired electrons exist in this complex, and what is its crystal field stabilization energy (CFSE) in terms of the octahedral splitting parameter Δ_o ?

- (A) 0 unpaired electrons, $\text{CFSE} = -2.4 \Delta_o$
- (B) 4 unpaired electrons, $\text{CFSE} = -0.4 \Delta_o$
- (C) 0 unpaired electrons, $\text{CFSE} = -1.2 \Delta_o$
- (D) 2 unpaired electrons, $\text{CFSE} = -1.6 \Delta_o$
- (E) 1 unpaired electron, $\text{CFSE} = -2.0 \Delta_o$

Astrophysics (Questions 16–20)

16. Star A has a surface effective temperature $T_A = 12000$ K and radius $R_A = 2R_\odot$. Star B has a surface effective temperature $T_B = 6000$ K and radius $R_B = 8R_\odot$. What is the ratio of the bolometric luminosity of Star A to Star B (L_A/L_B)?
- (A) 0.25
 - (B) 0.50
 - (C) 1.00
 - (D) 2.00
 - (E) 4.00
17. A spectroscopic binary consists of two stars in circular orbits around their common center of mass, viewed edge-on (inclination $i = 90^\circ$). The observed maximum radial velocity amplitudes of the two components are $K_1 = 30$ km/s and $K_2 = 60$ km/s, and the orbital period is $T = 3.14 \times 10^6$ s (≈ 36.3 days). What is the total mass ($M_1 + M_2$) of the binary system in solar masses ($M_\odot$)?
- (A) $0.85 M_\odot$
 - (B) $1.42 M_\odot$
 - (C) $2.68 M_\odot$
 - (D) $4.26 M_\odot$
 - (E) $7.85 M_\odot$
18. An optical emission line with rest-frame wavelength $\lambda_0 = 656.3$ nm ($\text{H}\alpha$) is observed in the spectrum of a distant quasar at $\lambda_{\text{obs}} = 1968.9$ nm. Assuming standard cosmological redshift $z = \Delta\lambda/\lambda_0$, what was the cosmological scale factor $a(t)$ of the Universe when this light was emitted, normalized to $a(t_0) = 1$ today?
- (A) $\frac{1}{2}$
 - (B) $\frac{1}{3}$
 - (C) $\frac{1}{4}$
 - (D) $\frac{2}{3}$
 - (E) $\frac{1}{\sqrt{3}}$

19. The Schwarzschild radius of a supermassive black hole is measured to be $R_s = 1.20 \times 10^{10}$ m. What is the radius of its Innermost Stable Circular Orbit (ISCO) for test particles orbiting around this non-rotating (Schwarzschild) black hole?
- (A) 6.00×10^9 m
 (B) 1.20×10^{10} m
 (C) 1.80×10^{10} m
 (D) 3.60×10^{10} m
 (E) 4.80×10^{10} m
20. A main-sequence star has mass $M = 5.0 M_\odot$. Assuming the mass-luminosity relation for intermediate-mass stars is roughly $L \propto M^{3.5}$ and the main-sequence lifetime of the Sun is $t_\odot \approx 10^{10}$ years, estimate the main-sequence lifetime of this $5.0 M_\odot$ star.
- (A) 1.8×10^7 yr
 (B) 8.9×10^7 yr
 (C) 3.2×10^8 yr
 (D) 7.5×10^8 yr
 (E) 1.2×10^9 yr

Artificial Intelligence & Machine Learning (Questions 21–25)

21. Consider training a binary classification model using cross-entropy loss:

$$L(w) = -y \ln \sigma(w^T x) - (1 - y) \ln(1 - \sigma(w^T x))$$

where $\sigma(z) = \frac{1}{1+e^{-z}}$ is the sigmoid activation function and $\hat{y} = \sigma(w^T x)$. What is the analytical gradient of the loss function with respect to the weight vector w ?

- (A) $\nabla_w L = (\hat{y} - y)x$
 (B) $\nabla_w L = (\hat{y} - y)\hat{y}(1 - \hat{y})x$
 (C) $\nabla_w L = \frac{\hat{y} - y}{\hat{y}(1 - \hat{y})}x$
 (D) $\nabla_w L = 2(\hat{y} - y)x$
 (E) $\nabla_w L = (y - \hat{y})x$
22. In deep Convolutional Neural Networks, an input tensor of spatial dimensions 64×64 with 16 channels is passed through a convolutional layer with 32 filters of size 5×5 , padding $p = 2$, and stride $s = 2$. What is the resulting spatial shape (height $\times$ width $\times$ channels) of the output feature map?
- (A) $30 \times 30 \times 32$
 (B) $32 \times 32 \times 32$
 (C) $32 \times 32 \times 16$

(D) $64 \times 64 \times 32$

(E) $16 \times 16 \times 32$

23. In PyTorch, what will the following Python script print?

```
import torch

x = torch.tensor([2.0, 3.0], requires_grad=True)
y = x[0]**2 + 3 * x[1]**3
y.backward()
print(x.grad.tolist())
```

(A) [4.0, 9.0]

(B) [4.0, 27.0]

(C) [4.0, 81.0]

(D) [2.0, 27.0]

(E) RuntimeError: grad can be implicitly created only for scalar outputs

24. In transformer-based large language models, the Scaled Dot-Product Attention is computed as:

$$\text{Attention}(Q, K, V) = \text{softmax}\left(\frac{QK^T}{\sqrt{d_k}}\right)V$$

Why is the dot-product matrix divided by the scaling factor $\sqrt{d_k}$?

(A) To reduce the asymptotic time complexity from $O(N^2)$ to $O(N\sqrt{d_k})$.

(B) To ensure that the queries and keys have zero mean.

(C) To prevent the dot products from growing excessively large for high dimensions, which would push the softmax function into regions with vanishingly small gradients.

(D) To project keys and queries onto a bounded unit sphere before matrix multiplication.

(E) To eliminate the need for causal masking during auto-regressive generation.

25. Consider the Principal Component Analysis (PCA) dimensionality reduction technique applied to a centered data matrix $X \in \mathbb{R}^{N \times D}$. The first principal component vector $v_1 \in \mathbb{R}^D$ is defined as the unit vector ($\|v_1\|_2 = 1$) that maximizes what quantity?

(A) The reconstruction mean squared error $\|X - Xv_1v_1^T\|_F^2$

(B) The empirical variance of the projected data points: $\frac{1}{N}v_1^T X^T X v_1$

(C) The L_1 sparsity norm $\|v_1\|_1$

(D) The mutual information between feature columns

(E) The trace of the pseudo-inverse $(X^T X)^+$

Section II: Free Response Questions (Questions 26–30)

Scoring reminder: Numerical tolerance is $\pm 3.0\%$ of the official reference answer.

You may use plus, minus sign, decimal point, exponential form e or E, and numbers in your submitted answer. For example, the following numerical answers are accepted: 3.14159, 0.314159e1, 31.4159e-2, -1.6e-19, -1.6E-19, -0.16E-18

No other alphabetical letter or symbol is accepted in the numerical answer. Fractions are not accepted. You will NOT need to specify any unit for FRQ problem.

26. [Mathematics]

Find the exact real volume of the solid bounded by the elliptic paraboloid $z = 4 - 2x^2 - y^2$ and the xy -plane ($z = 0$). State your numerical answer rounded to three significant figures.

27. [Physics]

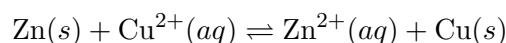
A thin, uniform circular wire loop of radius $r = 0.150$ m and electrical resistance $R = 2.50 \, \Omega$ lies flat in the xy -plane. A spatially uniform but time-dependent magnetic field is applied in the $+z$ direction given by:

$$B(t) = 0.800 \sin(120\pi t) \quad (\text{Tesla})$$

Calculate the root-mean-square (RMS) electric current, I_{rms} , induced in the loop in Amperes.

28. [Chemistry]

An electrochemical cell is set up at 298.15 K based on the standard Daniell cell:



The standard reduction potentials are $E^\circ(\text{Cu}^{2+}/\text{Cu}) = +0.340$ V and $E^\circ(\text{Zn}^{2+}/\text{Zn}) = -0.763$ V. The initial concentrations are $[\text{Cu}^{2+}] = 0.0100$ M and $[\text{Zn}^{2+}] = 1.500$ M. Calculate the initial cell potential E_{cell} in Volts.

29. [Astrophysics]

A spherical, non-rotating interstellar gas cloud of mass $M = 2.00 \times 10^{31}$ kg ($10.05 M_\odot$) and uniform temperature $T = 20.0$ K is composed of pure molecular hydrogen (H_2 , mean molecular weight $\mu = 2.016$ g/mol). The Jeans mass for a uniform spherical cloud is given by:

$$M_J = \left(\frac{5k_B T}{G \mu m_H} \right)^{3/2} \left(\frac{3}{4\pi \rho} \right)^{1/2}$$

where $m_H = 1.6605 \times 10^{-27}$ kg and $k_B = 1.3807 \times 10^{-23}$ J/K. Calculate the critical threshold mass density ρ_c in kg/m^3 such that the cloud mass equals the Jeans mass ($M = M_J$).

30. [Artificial Intelligence / Deep Learning]

A multi-layer perceptron (MLP) takes an input vector $x \in \mathbb{R}^{128}$ and passes it through three successive fully-connected linear layers:

- Layer 1: $128 \rightarrow 256$ units, followed by a ReLU activation.
- Layer 2: $256 \rightarrow 64$ units, followed by Layer Normalization (with learnable scale γ and shift β per feature).
- Layer 3: $64 \rightarrow 10$ units, with no activation (output logits).

Every linear layer includes additive bias parameters. Calculate the total number of trainable parameters in this entire network.